Math 250A Lecture 5 Notes

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September 7, 2017

1 Symmetric Groups

1.1 Basic definitions

Definition 1.1. The *symmetric group* S_n is the group of all permutations of the n points $\{1, \ldots, n\}$.

$|S_n| = n!$ because there are n choices for the image of 1, then $n-1$ choices for the image of 2, etc. We denote elements using cycle notation: $(a\,b\,c\,d)$ is the function taking $a \mapsto b \mapsto c \mapsto d$.

Definition 1.2. A transposition is a permutation that exchanges 2 elements and fixes all others.

Proposition 1.1. *S_n is generated by the transpositions* $(1\,2), (2\,3), (3\,4), \ldots, (n-1\,n)$.

Proof. This is "bubblesort," the "2nd worst" sorting algorithm.[1] In the worst case, bubblesort takes $n(n-1)/2$ exchanges to sort a list of n elements. □

1.2 The alternating group

Look at S_n acting on variables $x_1, \ldots, x_n$. It also acts on $\mathbb{C}[x_1, \ldots, x_n]$, polynomials in n variables. Look at the discriminant,

$$\Delta = (x_1 - x_2)(x_2 - x_3) \cdots (x_1 - x_3) \cdots (x_{n-1-x_n}) = \prod_{i<j} (x_i - x_j).$$

Any $\sigma \in S_n$ maps Δ to Δ or $-\Delta$, so there exists some homomorphism $\varepsilon : S_n \to \{\pm 1\}$.

Definition 1.3. The *alternating group* A_n is the subgroup of elements $\sigma \in S_n$ such that $\sigma(\Delta) = \Delta$); this is the kernel of ε.

So A_n is normall in S_n, of order $n!/2$.

[1] The "worst" algorithm is called bogosort.

1.3 S_n, A_n, and platonic solids

Symmetries of platonic solids are very closely related to the groups S_n and A_n.

The rotations and reflections of a tetrahedron is S_4, acting on the vertices; the rotations are then A_4. The rotations of a cube or an octahedron are given by S_4 acting on the permutations of the diagonals; then the rotations and reflections are given by $S_4 \times \mathbb{Z}/2\mathbb{Z}$. The number of rotations of a dodecahedron or an icosahedron is given by permutations of the five inscribed "inner cubes," which gives a homomorphism of rotations to S_5, and this group is A_5; then the rotations and reflections are given by $A_5 \times \mathbb{Z}/2\mathbb{Z}$.

We summarize the results in this table:

platonic solid	number of rotations	number of rotations and reflections
tetrahedron	12 (A_4)	24 (S_4)
cube/octahedron	24 (S_4)	48 ($S_4 \times \mathbb{Z}/2\mathbb{Z}$)
dodecahedron/icosahedron	60 (A_5)	120 ($A_5 \times \mathbb{Z}/2\mathbb{Z} \ncong S_5$)

These groups are "spherical reflection groups."

1.4 Conjugacy classes of S_n

We can write any element of S_n as a product of disjoint cycles.

Definition 1.4. The cycle shape is the sizes of the cycles with multiplicities.

Example 1.1. The permutation $(1\,2\,4)(5\,7\,8)(6\,9)(10)(3)$ has cycle shape $3^2, 2, 1^2$.

Two elements are conjugate if they have the same cycle shape. Given a, b, with the same cycle shape, how can we find g with $a = gbg^{-1}$? Write out the two permutations in cycle notation and pair off elements:

$$(1\,2\,4)(5\,7\,8)(6\,9)(10)(3)$$

$$\uparrow\uparrow\uparrow\ \ \uparrow\uparrow\uparrow\ \ \uparrow\uparrow\ \ \ \uparrow\ \ \uparrow$$

$$(2\,4\,5)(6\,7\,8)(1\,3)(9)(10)$$

This gives us $g = (1\,6\,5\,4\,2\,1)(3\,9\,10)(7)(8)$.

Example 1.2. How many conjugacy classes are the of S_4? This is the number of cycle shapes, which is also the number of partitions of 4. Denoting C_σ as the conjugacy class (viewing S_4 as the rotations of a cube) and G_σ as the stabilizer under the action of conjugation (also is the centralizer).

partition	cycle shape	C_σ	$\lvert G_\sigma\rvert$	$\lvert C_\sigma\rvert = \lvert G\rvert / \lvert G_\sigma\rvert$
1+1+1+1	1^4	identity	24	1
2+1+1	$2, 1^2$	rotation by π	4	6
3+1	$3, 1$	rotation by $2\pi/3$	3	8
2+2	2^2	rotation by π	8	3
4	4	rotation by $\pi/2$	4	6

If σ has cycle shape $1^{n_1}2^{n_2}3^{n_3}\cdots$, then the number of elements in the centralizer is $1^{n_1}n_1! \cdot 2^{n_2}n_2! \cdot \cdots$.

1.5 Normal subgroups of S_n

What are the normal subgroups of S_n? We already know of $\{e\}$, A_n, and S_n. Viewing S_4 as the rotations of a cube, we have that S_4 acts on 3 lines by permuting them; so we have a homomorphism $S_4 \to S_3$, where the kernel is a normal subgroup of order 4 (the identity + 3 rotations by π). Following this pattern, we have homomorphisms S_2 onto S_1, S_3 onto S_2, and S_4 onto S_3. However, the pattern breaks because there is no homomorphism from S_5 onto S_4; S_5 has a simple subgroup A_5, the rotations of an icosahedron. If N is any normal subgroup of S_5, $N \cap A_5$ is normal in A_5, so it is 1 or 5. So the only normal subgroups of S_5 are $\{e\}$, A_5, and S_5.

Theorem 1.1. *A_n is simple for $n \geq 5$.*

Proof. We sketch a proof using induction on n. Suppose N is normal in S_n. Pick an element $g \in N$ with $g \neq e$. Find h so that $ghg^{-1}h^{-1}$ fixes the point 1 (exercise). So $ghg^{-1}h^{-1} = g(hg^{-1}h^{-1})$ is also in N, which makes N have nontrivial intersection with S_{n-1} (things fixing 1). So $N \cap S_{n-1} = A_{n-1}$ or S_{n-1}. So N contains all elements of A_n fixing 1. Similarly, it contains all elements fixing i for any i. These generate A_n (also an exercise). □

Example 1.3. There are three groups of order 120 containing A_5 and $\mathbb{Z}/2\mathbb{Z}$ as composition factors.

1. $A_5 \times \mathbb{Z}/2\mathbb{Z}$
2. S_5, which has a subgroup A_5 and the quotient group $\mathbb{Z}/2\mathbb{Z}$
3. Binary icosahedral group[2], which has a quotient group A_5 and a subgroup $\mathbb{Z}/2\mathbb{Z}$

[2] Let G be this group. Then S^3/G, the cosets of G in S^3 (not a group), has the same homology as S^3 but is not homeomorphic to S_3.

1.6 Outer automorphisms of S_n

Conjugation is an automorphism of a group G, and we get an exact sequence

$$1 \to Z(G) \to \text{conjugations} \to \text{Aut}(G) \to \text{outer automorphisms} \to 1.$$

For $n \geq 3$ with $n \neq 6$, $\text{Aut}(S_n) \cong S_n$, and all these automorphisms are inner automorphisms.

Let's find a non-inner automorphism of S_6. Start with S_5. This has a subgroup of order 20. S_5 acts on $0, 1, 2, 3, 4 \in \mathbb{F}_5$, and has the following subgroup: all permutations of the form $x \mapsto ax + b$ for $a, b \in \mathbb{F}_5$. So S_5 has a subgroup of index 6, so it acts transitively on 6 points, giving us a homomorphism from $S_5 \to S_6$ which is different from the usual such homomorphisms that fix some element (which are not transitive). S_6 has 12 subgroups $\cong S_5$, not 6, as we might expect.

Any subgroup of index n in G products a homomorphism from $G \to S_n$, where G acts transitively on n points, so any subgroup of index 6 in S_6 gives a homomorphism from $S_6 \to S_6$. Pick one of the "funny" homomorphisms $S_5 \to S_6$ to get a homomorphism from $S_6 \to S_6$. Check that this is not an inner automorphism (exercise).